

Finite Difference Distributions for Ginibre Ensemble

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Abstract

The Ginibre ensemble of complex random matrices is studied. The complex valued random variable of second difference of complex energy levels is defined. For the $N=3$ dimensional ensemble are calculated distributions of second difference, of real and imaginary parts of second difference, as well as of its radius and of its argument (angle). For the generic N -dimensional Ginibre ensemble an exact analytical formula for second difference's distribution is derived. The comparison with real valued random variable of second difference of adjacent real valued energy levels for Gaussian orthogonal, unitary, and symplectic, ensemble of random matrices as well as for Poisson ensemble is provided.

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I. INTRODUCTION

Random Matrix theory assumes that the Hamiltonian operator H of a generic quantum system is unknown and unknowable [1–3]. The matrix elements H_{ij} of Hamiltonian in given basis of Hilbert space are random variables. Their distributions are given by appropriate formulae depending on studied Random Matrix ensemble [1–3]. The symmetry properties of H which is hermitean lead us to Gaussian ensembles of random matrices: orthogonal GOE, unitary GUE, symplectic GSE, as well as to circular ensembles: orthogonal COE, unitary CUE, and symplectic CSE. The energies E_i of quantum systems calculated from diagonalization of Hamiltonian matrix H_{ij} are random variables with appropriate distributions and they exhibit generic classes of level repulsion. It was Wigner who firstly discovered level repulsion phenomenon [1–3]. The applications of Random Matrix theory are very broad: nuclear physics (slow neutron resonances, highly excited complex nuclei), condensed phase physics (fine metallic particles, random Ising model [spin glasses]), quantum chaos (quantum billiards, quantum dots), disordered mesoscopic systems (transport phenomena). J. Ginibre studied very general case of random Hamiltonians. He dropped the assumption of hermiticity of Hamiltonians and he considered generic complex valued matrices [1,2,4,5]. Thus, H 's belong to general linear Lie group $GL(N, \mathbf{C})$, where N is dimension and \mathbf{C} is complex numbers field. Therefore, the energies Z_i of quantum system ascribed to Ginibre ensemble are complex valued. This is an extension of Gaussian or circular ensembles. J. Ginibre postulated the following joint probability density function of random vector of complex eigenvalues Z_1, \dots, Z_N for $N \times N$ Hamiltonian matrices [1,2,4,5]:

$$P(z_1, \dots, z_N) = \prod_{j=1}^N \frac{1}{\pi \cdot j!} \cdot \prod_{i < j}^N |z_i - z_j|^2 \cdot \exp\left(-\sum_{j=1}^N |z_j|^2\right). \quad (1)$$

We emphasise that Z_i 's are *complex valued* random variables, and z_i 's are *complex* sample points ($z_i \in \mathbf{C}$).

One must emphasise here Wigner and Dyson's electrostatic analogy. A Coulomb gas of N unit charges moving on complex plane (Gauss' plane) \mathbf{C} is considered. The vectors of positions of charges are z_i 's and potential energy of the system is:

$$U(z_1, \dots, z_N) = -\sum_{i < j} \ln |z_i - z_j| + \frac{1}{2} \sum_i |z_i|^2. \quad (2)$$

If gas is in thermodynamical equilibrium at temperature $T = \frac{1}{2k_B}$ ($\beta = \frac{1}{k_B T} = 2$, k_B is Boltzmann's constant), then probability density function of vectors of positions is $P(z_1, \dots, z_N)$ Eq. (1). Thus, complex energies of quantum system and vectors of positions of charges of Coulomb gas are analogous to each other. In view of above analogy one can consider the complex spacings $\Delta^1 Z_i$ of complex energies of quantum system:

$$\Delta^1 Z_i = Z_{i+1} - Z_i, i = 1, \dots, (N - 1), \quad (3)$$

as vectors of relative positions of electric charges of Coulomb gas. For Ginibre ensemble it were calculated the distributions of *real valued* absolute values of spacings of nearest neighbour *ordered* energies. We complement this by introduction of complex valued second differences $\Delta^2 Z_i$ of complex energies:

$$\Delta^2 Z_i = Z_{i+2} - 2Z_{i+1} + Z_i, i = 1, \dots, (N - 2). \quad (4)$$

The second differences are three energy level magnitudes that enhance our knowledge of quantum systems with complex energies. Moreover, $\Delta^2 Z_i$'s can be regarded as vectors of relative positions of vectors of relative positions of electric charges. One can observe movement of electric charges in Cartesian frame of references or in polar one. Since the two-dimensional vectors in Cartesian frame of reference have their projections on co-ordinate axes, therefore the real and imaginary parts of $\Delta^2 Z_i$'s, namely $\text{Re}\Delta^2 Z_i$, $\text{Im}\Delta^2 Z_i$, can be interpreted as projections of second differences on abscissa and ordinate axes, respectively. The radii $|\Delta^2 Z_i|$, and arguments (angles) $\text{Arg}\Delta^2 Z_i$ of second differences have interpretations of polar co-ordinates of $\Delta^2 Z_i$ vectors. $\Delta^2 Z_i$'s are analogous to real valued second differences:

$$\Delta^2 E_i = E_{i+2} - 2E_{i+1} + E_i, i = 1, \dots, (N - 2), \quad (5)$$

of adjacent ordered increasingly real valued energies E_i defined for GOE, GUE, GSE, and Poisson ensemble PE (where Poisson ensemble is composed of uncorrelated randomly distributed energies) [6–9]. We will calculate the distributions of $\Delta^2 Z_i$, $\text{Re}\Delta^2 Z_i$, $\text{Im}\Delta^2 Z_i$, $|\Delta^2 Z_i|$, $\text{Arg}\Delta^2 Z_i$. Finally, we will compare these results with second difference distributions for Gaussian ensembles, and Poisson ensemble [1,10–15].

II. SECOND DIFFERENCE DISTRIBUTIONS

We use formula (1) with $N = 3$ and define the following complex valued random vector (Y_1, Y_2, Y_3) and real A_j and imaginary B_j parts:

$$Y_1 = \Delta^2 Z_1, Y_2 = Z_2 - Z_3, Y_3 = Z_3, Y_j = (A_j, B_j), A_j = \text{Re}Y_j, B_j = \text{Im}Y_j, j = 1, \dots, 3. \quad (6)$$

The change of the variable formula gives us the result for joint probability density function of random vector (Y_1, Y_2, Y_3) [16]:

$$\begin{aligned} f_{(Y_1, Y_2, Y_3)}(y_1, y_2, y_3) &= f_{(A_1, B_1, A_2, B_2, A_3, B_3)}(a_1, b_1, a_2, b_2, a_3, b_3) = \\ &= \frac{1}{12\pi^3} \cdot [(a_1 + a_2)^2 + (b_1 + b_2)^2] \cdot [a_2^2 + b_2^2] \cdot [(a_1 + 2a_2)^2 + (b_1 + 2b_2)^2] \cdot \\ &\cdot [\exp(-(a_1 + 2a_2 + a_3)^2 - (b_1 + 2b_2 + b_3)^2 - (a_2 + a_3)^2 - (b_2 + b_3)^2 - a_3^2 - b_3^2)], \end{aligned} \quad (7)$$

where $y_j = (a_j, b_j) \in \mathbf{C}$ are complex random sample points. [In order to obtain Eq. (7) we used the complex valued linear map

$$(Y_1, Y_2, Y_3) = \Xi(Z_1, Z_2, Z_3), \Xi = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad (8)$$

where Jacobian of the inverse map Ξ^{-1} is equal to unity $\text{Jac}(\Xi^{-1}) = 1$. We integrate out Y_3 :

$$f_{(Y_1, Y_2)}(y_1, y_2) = \int_{\mathbf{C}} f_{(Y_1, Y_2, Y_3)}(y_1, y_2, y_3) dy_3, \quad (9)$$

and we obtain following marginal probability density function:

$$\begin{aligned} f_{(Y_1, Y_2)}(y_1, y_2) &= f_{(A_1, B_1, A_2, B_2)}(a_1, b_1, a_2, b_2) = \\ &= \frac{1}{36\pi^2} [(a_1 + a_2)^2 + (b_1 + b_2)^2] [a_2^2 + b_2^2] [(a_1 + 2a_2)^2 + (b_1 + 2b_2)^2] \cdot \\ &\cdot \exp\left[-\frac{2}{3}a_1^2 - 2a_1a_2 - 2a_2^2 - \frac{2}{3}b_1^2 - 2b_1b_2 - 2b_2^2\right]. \end{aligned} \quad (10)$$

Now we calculate the marginal probability density function of second difference:

$$f_{Y_1}(y_1) = f_{(A_1, B_1)}(a_1, b_1) = \int_{\mathbf{C}} f_{(Y_1, Y_2)}(y_1, y_2) dy_2 = \frac{1}{576\pi} [(a_1^2 + b_1^2)^2 + 24] \cdot \exp\left(-\frac{1}{6}(a_1^2 + b_1^2)\right). \quad (11)$$

Now we derive the distributions of real part A_1 and of imaginary part B_1 of second difference:

$$f_{A_1}(a_1) = \int_{\mathbf{R}} f_{(A_1, B_1)}(a_1, b_1) db_1 = \frac{\sqrt{6}}{576\sqrt{\pi}} (a_1^4 + 6a_1^2 + 51) \cdot \exp\left(-\frac{1}{6}a_1^2\right), \quad (12)$$

$$f_{B_1}(b_1) = \int_{\mathbf{R}} f_{(A_1, B_1)}(a_1, b_1) da_1 = \frac{\sqrt{6}}{576\sqrt{\pi}} (b_1^4 + 6b_1^2 + 51) \cdot \exp\left(-\frac{1}{6}b_1^2\right), \quad (13)$$

where \mathbf{R} is field of real numbers.

We transform complex valued random variable of second difference Y_1 to polar coordinates' variables R_1, Φ_1 :

$$R_1 = |Y_1|, \Phi_1 = \text{Arg} Y_1, \quad (14)$$

and we obtain by standard method the following probability density function of random vector (R_1, Φ_1) :

$$f_{(R_1, \Phi_1)}(r_1, \phi_1) = \Theta(r_1) \frac{1}{576\pi} r_1 (r_1^4 + 24) \cdot \exp\left(-\frac{1}{6}r_1^2\right) \quad (15)$$

(the Jacobian of transformation is r_1 , Θ is Heaviside (step) function [16]). It follows that:

$$f_{R_1}(r_1) = \Theta(r_1) \frac{1}{288} r_1 (r_1^4 + 24) \cdot \exp\left(-\frac{1}{6}r_1^2\right), f_{\Phi_1}(\phi_1) = \frac{1}{2\pi}, \phi_1 \in [0, 2\pi]. \quad (16)$$

III. N-DIMENSIONAL GINIBRE ENSEMBLE

The case of generic N-dimensional Ginibre ensemble is of special physical interest ($N \geq 3$). We will calculate the distribution of second difference for the ensemble. One commences with n-level correlation function:

$$P_n(z_1, \dots, z_n) = \int_{\mathbf{C}^{N-n}} P(z_1, \dots, z_N) dz_{n+1} \dots dz_N = \quad (17)$$

$$= \pi^{-n} \exp\left(-\sum_{i=1}^n |z_i|^2\right) \det D^{(n)}, \quad (18)$$

$$D_{ij}^{(n)} = e_{N-1}(z_i z_j^*), i, j = 1, \dots, n, e_{N-1}(z) = \sum_{k=0}^{N-1} \frac{z^k}{k!}, \quad (19)$$

Ref. [1]. In order to calculate the distribution of complex second difference $W_1 = \Delta^2 Z_1$ for N-dimensional Ginibre ensemble one substitutes $n=3$ to Eq. (17) and defines random vector $W = (W_1, W_2, W_3)$:

$$(W_1, W_2, W_3) = \Omega(Z_1, Z_2, Z_3), \Omega = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (20)$$

The probability density function of random vector W reads:

$$P_3(w_1, w_2, w_3) = P_3(\Omega^{-1}(w_1, w_2, w_3)) = P_3(w_1 + 2w_2 - w_3, w_2, w_3), \quad (21)$$

Ref. [16]. Hence, the distribution of second difference is:

$$P_3(w_1) = \int_{\mathbf{C}^2} P_3(w_1 + 2w_2 - w_3, w_2, w_3) dw_2 dw_3. \quad (22)$$

We combine Eqs (17), (18), (19), (21), (22), and we use Laplace's expansion of determinant $\det D^{(n)}$:

$$P_3(w_1) = \pi^{-3} \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \int_{\mathbf{C}^2} \exp\left(-\sum_{i=1}^3 |(\Omega^{-1}w)_i|^2\right) \prod_{k=1}^3 e_{N-1}[(\Omega^{-1}w)_k \cdot (\Omega^{-1}w)_{\mathcal{P}k}^*] dw_2 dw_3, \quad (23)$$

where \mathcal{P} is permutation of indices $(1, 2, 3)$. The only nonzero contribution to Eq. (23) is for identity permutation $\mathcal{P} = \text{id} = (1, 2, 3)$. It results from the fact that other permutations produce factors that are periodic functions of arguments $\text{Arg}w_2, \text{Arg}w_3$ of complex numbers w_2, w_3 (the integrals over w_2, w_3 can be transformed to polar co-ordinates where the arguments $\text{Arg}w_2, \text{Arg}w_3$ are integrated over $[0, 2\pi]$). Hence,

$$P_3(w_1) = \pi^{-3} \int_{\mathbf{C}^2} \exp\left(-\sum_{i=1}^3 |(\Omega^{-1}w)_i|^2\right) \prod_{k=1}^3 e_{N-1}[|(\Omega^{-1}w)_k|^2] dw_2 dw_3, \quad (24)$$

Therefore, considering Eq. (19) one has:

$$P_3(w_1) = \pi^{-3} \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} \sum_{j_3=0}^{N-1} \frac{1}{j_1! j_2! j_3!} I_{j_1 j_2 j_3}(w_1), \quad (25)$$

$$I_{j_1 j_2 j_3}(w_1) = \int_{\mathbf{C}^2} \exp\left(-\sum_{i=1}^3 |(\Omega^{-1}w)_i|^2\right) \prod_{k=1}^3 |(\Omega^{-1}w)_k|^{2j_k} dw_2 dw_3. \quad (26)$$

One changes variables in Eq. (26) in following way: $V_2 = 2W_2, V_3 = -W_3$, and obtains:

$$I_{j_1 j_2 j_3}(w_1) = 2^{-2j_2} \int_{\mathbf{C}^2} \exp(-|w_1 + v_2 + v_3|^2 - \frac{1}{4}|v_2|^2 - |v_3|^2) |w_1 + v_2 + v_3|^{2j_1} |v_2|^{2j_2} |v_3|^{2j_3} dv_2 dv_3. \quad (27)$$

The above double integral can be calculated by extending the exponent by additional terms proportional to λ_i parameters and considering appropriate derivatives:

$$I_{j_1 j_2 j_3}(w_1) = 2^{-2j_2} \frac{\partial^{j_1+j_2+j_3}}{\partial^{j_1} \lambda_1 \partial^{j_2} \lambda_2 \partial^{j_3} \lambda_3} F(w_1, \lambda_1, \lambda_2, \lambda_3)|_{\lambda_i=0}, \quad (28)$$

$$F(w_1, \lambda_1, \lambda_2, \lambda_3) = \int_{\mathbf{C}^2} \exp[G(w_1, v_2, v_3, \lambda_1, \lambda_2, \lambda_3)] dv_2 dv_3, \quad (29)$$

$$G(w_1, v_2, v_3, \lambda_1, \lambda_2, \lambda_3) = (\lambda_1 - 1)|w_1 + v_2 + v_3|^2 + (\lambda_2 - \frac{1}{4})|v_2|^2 + (\lambda_3 - 1)|v_3|^2. \quad (30)$$

Finally, we derive $F(w_1, \lambda_1, \lambda_2, \lambda_3)$ by transformation of parametric quadratic form $G(w_1, v_2, v_3, \lambda_1, \lambda_2, \lambda_3)$ to canonical form and integrating over v_2, v_3 :

$$F(w_1, \lambda_1, \lambda_2, \lambda_3) = A(\lambda_1, \lambda_2, \lambda_3) \exp[-B(\lambda_1, \lambda_2, \lambda_3)|w_1|^2], \quad (31)$$

where

$$A(\lambda_1, \lambda_2, \lambda_3) = \frac{(2\pi)^2}{(\lambda_1 + \lambda_2 - \frac{5}{4}) \cdot (\lambda_1 + \lambda_3 - \frac{5}{4}) - (\lambda_1 - 1)^2}, \quad (32)$$

$$B(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1 - 1) \cdot \frac{2\lambda_1 - \lambda_2 - \lambda_3 + \frac{1}{2}}{2\lambda_1 + \lambda_2 + \lambda_3 - \frac{9}{2}}. \quad (33)$$

Hence, we obtained analytical formula for distribution $P_3(w_1)$ of second difference for N-dimensional Ginibre ensemble combining Eqs. (25), (28), (31), (32), (33). What is worth to be mentioned is that second difference's distribution is a triple sum of zero-centred Gaussian distributions with different widths. The distribution is again function of only modulus $|w_1|$ of second difference and it has global maximum at origin.

IV. COMPARISON

Finally, in order to compare our results for second difference for Ginibre ensemble with previous ones for Gaussian and Poissonian ensembles we must rescale them by division by appropriate magnitude. We consider such rescaled dimensionless second differences in following way. The mean values of second differences either in real or in complex case are zero, hence we cannot divide second differences by the mean values. It follows that we divide real valued second differences $\Delta^2 E_1$ for GOE(3) ($\beta = 1$), for GUE(3) ($\beta = 2$), for GSE(3) ($\beta = 4$), for PE ($\beta = 0$), by mean spacings $\langle S_\beta \rangle$ calculated for those ensembles, and we create new dimensionless second differences:

$$C_\beta = \frac{\Delta^2 E_1}{\langle S_\beta \rangle}, \quad (34)$$

respectively [6–9]. The probability density functions of C_β 's were calculated for GOE, GUE, GSE, and PE [6–9]. Since second difference for Ginibre ensemble is complex valued, then we

choose its real part A_1 for comparison with C_β 's. One divides A_1 by analogue of $\langle S_\beta \rangle$'s, which is mean value $\langle R_1 \rangle$ of radius R_1 Eq. (16) of $\Delta^2 Z_1$:

$$\langle R_1 \rangle = \int_0^\infty r_1 f_{R_1}(r_1) dr_1 = \frac{53}{64} \sqrt{6\pi}. \quad (35)$$

Hence:

$$X_1 = \frac{A_1}{\langle R_1 \rangle}, \quad (36)$$

is rescaled dimensionless A_1 . The probability distributions of C_β 's and of X_1 depend on the same real variable x which is equal to $\frac{\Delta^2 e_1}{\langle S_\beta \rangle}$, $\frac{a_1}{\langle R_1 \rangle}$, respectively (e_1 is value of energy E_1).

The second differences' distributions for Gaussian, Poisson, and Ginibre ensembles assume global maxima at origin and that they are unimodular. Firstly, it extends the theorem of level homogenisation to Ginibre ensemble [6–9]. We can formulate the following law: *Energy levels for Gaussian ensembles, for Poisson ensemble, and for Ginibre ensemble tend to be homogeneously distributed.* The second differences' distributions assume global maxima at origin no matter whether second differences are real or complex. From Coulomb gas' point of view it is easier to be understood. The unit charges behave in such a way that the vectors of relative positions of charges statistically tend to be zero. It could be called stabilisation of structure of system of electric charges. The above results can be extended to study of higher differences' distributions for Ginibre ensemble.

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